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Approximation by Minimum Norm Interpolants in the Disc Algebra*

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1. INTRODUCTION AND RESULTS

Most problems in approximation theory deal with approximating functions in an infinite dimensional normed linear function space using an increasing sequence of finite dimensional subspaces. However, it is also of interest and importance to consider the problem of approximation using subspaces of finite (or even small) codimension.

Let X be a normed linear function space and let $M_N \subset X$ be an ideal corresponding to functions which vanish on a "small" set E_N. Let $m_N(f)$ denote a best approximant (if it exists) to f from M_N. Hence, $S_N(f) = f - m_N(f)$ interpolates f on E_N and will be called a minimal norm interpolant of f. If the sets E_i satisfy

$$E_1 \subset E_2 \subset \cdots, \tag{1.1}$$

then the study of the rate of decrease of $\|f - s_N(f)\| = \|m_N(f)\|$ is related to the problem of convergence of the interpolants of f with minimal norm. Such questions have been previously considered in the setting of Sobolev spaces (cf. [1, 2, 4, 5]).

In this note the rate of convergence of minimal norm interpolants is studied in the case where X is the disc algebra A. For this situation, the non-trivial ideals considered correspond to the subspaces of functions which vanish on a closed set E_N of measure zero on the boundary T of the unit disc D in the complex plane. Our main result is the following.

THEOREM. *Let $f \in A$ and let E_N, $N = 1, 2,\ldots,$ be closed subsets of T of measure zero such that*

$$\gamma_N = \log(\|f\|_T / \|f\|_{E_N}) \to 0. \tag{1.2}$$

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Then there exist minimal norm interpolants s_N of f satisfying $f - s_N \in M_N \equiv \{g \in A: g(E_N) = 0\}$ such that

$$\|f - s_N\| = O(\gamma_N). \tag{1.3}$$

Moreover, this rate of convergence is sharp in the sense that $O(\gamma_N)$ cannot be replaced by $o(\gamma_N)$.

In certain function spaces the rate of convergence is easily established. For instance, let $X = C[0, 1]$ and $M_N = \{f \in C[0, 1]: f(E_N) = 0\}$ where E_N are finite point sets in $[0, 1]$ satisfying (1.1). It is well known that M_N is a proximinal subspace and that $\text{dist}(f, M_N) = \|f|_{E_N}\|$. It is also easily seen that for each $f \in C[0, 1]$ there exists an $m_N \in M_N$ so that $\|f - s_N(f)\| = \|m_N\| = \|f|_{[0,1]\setminus E_N}\| - \|f|_{E_N}\| \equiv \gamma_N \geqslant 0$. Furthermore, if $T_N(f)$ is any other interpolant of f on E_N with minimal norm, then $\|f - T_N(f)\| \geqslant \gamma_N$.

Less obvious is the case of the disc algebra A. Unlike the case $C[0, 1]$, neither the proximinality of the ideal $M_N = \{g \in A: g|_{E_N} = 0\}$ nor the rate of convergence of minimal norm interpolants is clear although the former question is more easily handled than the latter. In fact from [3] and [6] the ideals in A described above are known to be M-ideals which in turn are known to be proximinal [8]. The answer to the latter question is less obvious and as the following example shows care should be exercised in selecting the minimum norm interpolants in order to guarantee convergence to f.

EXAMPLE. Let E_N denote the set of all 2^Nth roots of unity and $M_N = \{f \in A: f(E_N) = 0\}$. It is easily checked that $m_N = 1 - z^{2^N}$ is a best approximant to $f(z) \equiv 1$; however, $s_N = f - m_N = z^{2^N}$ does not converge to f in A.

In light of this example, the above theorem may be viewed as an existence theorem since it establishes a sequence of minimum norm interpolants which converges to f.

2. PROOF OF THEOREM

In what follows, E_N will denote a closed subset of measure zero of the unit circle T in the complex plane, A the disc algebra, M_N the subspace

$$M_N = \{g \in A: g|_{E_N} = 0\} \tag{2.1}$$

and $m_N = m_N(f)$ a best approximant to a given $f \in A$ from M_N. Also,

$\Delta(E_N, T)$ will designate the Hausdorff distance between E_N and T. Set

$$P_z(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(t)\, P_r(\theta - t)\, dt,$$
$$H_z(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(t)\,\frac{e^{it}+z}{e^{it}-z}\, dt$$

where $z = re^{i\theta}$ and $P_r(\theta - t) = \operatorname{Re}\{e^{it}+z)/(e^{it}-z)\}$, so that $P_z(w) = \operatorname{Re} H_z(w)$. We are now able to prove the main result in this note.

Choose any ρ_N such that $0 < \rho_N \to \infty$ and $\rho_N\gamma_N \to 0$. Hence $1 \leqslant (\|f\|_T/\|f\|_{E_N})^{\rho_N} \to 1$. Then it follows that

$$F_N(z) \equiv \rho_N \log(|f(z)|/\|f\|_{E_N}) < 1 \tag{2.2}$$

for all $z \in T$ and all large N, and $F_N(z) \leqslant 0$ if $z \in E_N$. Choose a real-valued function w_N on T such that

$$\begin{aligned} &w_N(z) = -\infty \text{ on } E_N,\ w_N(z) \to -\infty \text{ continuously as } z \to E_N,\\ &w_N(z) \leqslant -1 \text{ on } T,\\ &w_N(\cdot) \text{ is } C^1 \text{ on } T - E_N, \text{ and}\\ &w_N(\cdot) \in L^1(T). \end{aligned} \tag{2.3}$$

Hence, $-1 \leqslant 1/w_N(z) \leqslant 0$ holds for all z, and $w_N(z) = 0$ only if $z \in E_N$ and w_N is in $C^1(T\backslash E_N) \cap C(T)$.

Since $-F_N(z) \geqslant 0$ on E_N and $-F_N(z) > -1$ on T, we can choose w_N so that w_N satisfies the additional condition $1/w_N \leqslant -F_N$ on T. This implies $-1/w_N \geqslant F_N$ on T so that $-1/P_z(w_N) \geqslant F_N(z)$ on T and $-1/\rho_N P_z(w_N) \geqslant \log(|f(z)|/\|f\|_{E_N})$. Set $\phi_N(z) = e^{(1/\rho_N H_z(w_N))}$. Following the proof of Fatou's theorem (cf. [7, p. 80]), $1/H_z(w_N) \in A$, $\operatorname{Re} 1/H_z(w_N) \leqslant 0$, and $1/H_z(w_N)$ equals zero only on E_N. Hence $\phi_N \in A$, $|\phi_N| \leqslant 1$ and $\phi_N(z) = 1$ if and only if $z \in E_N$. Since ρ_N diverges to infinity. ϕ_N converges to 1 uniformly on T. Also, on T for large N,

$$\begin{aligned} |f(z)\,\phi_N(z)| &= |f(z)|\, e^{\operatorname{Re}(1/\rho_N H_z(w_N))} = |f(z)|\, e^{\operatorname{Re}(\overline{H_z(w_N)})/\rho_N|H_z(w_N)|^2)}\\ &= |f(z)|\, e^{P_z(w_N)/\rho_N|H_z(w_N)|^2} \leqslant |f(z)|\, e^{P_z(w_N)/\rho_N|P_z(w_N)|^2}\\ &= |f(z)\, e^{1/(\rho_N P_z(w_N))}| \leqslant \|f\|_{E_N}. \end{aligned} \tag{2.4}$$

Thus, setting $m_N = f(1 - \phi_N)$ the following properties are evident:

$$\begin{aligned} &m_N(z) = 0 \quad \text{if} \quad z \in E_N,\\ &|m_N(z) - f(z)| = |f(z)\,\phi_N(z)| \leqslant \|f\|_{E_N} \quad \text{for all} \quad z \in T, \quad \text{and}\\ &\|m_N\| \leqslant \|f\|\|\phi_N - 1\| \to 0 \quad \text{as} \quad N \to \infty. \end{aligned} \tag{2.5}$$

Hence, one deduces that for $s_N \equiv f - m_N$,

$$\begin{aligned}\|f - s_N\| &= O(\|\phi_N - 1\|) \\ &= O(\|e^{1/(\rho_N H_z(w_N))} - 1\|) \\ &= O\left(\frac{1}{\|\rho_N H_z(w_N)\|}\right).\end{aligned}$$

But since $1/(\|\rho_N H_z(w_N)\|) \leqslant 1/\|\rho_N P_z(w_N)\|$, $-\infty < w_N(z) < -1$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, dt = 1$$

we have $1/(\|\rho_N P_z(w_N)\|) < 1/\rho_N$ so that $\|f - s_N\| = O(1/\rho_N)$ for all ρ_N for which $\rho_N \gamma_N$ converges to 0. But an easy argument shows that actually $\|f - s_N\| = O(\gamma_N)$.

To prove the sharpness of (1.2), we assume that $\|f - s_N\| = O(r_N)$ where $r_N \to 0^+$ but $r_N^{-1}\|f - s_N\|$ is bounded away from zero. It must be shown that $\gamma_N = O(r_N)$. Pick N sufficiently large so that $\|f\|_{E_N} \geqslant \|f\|/2$. Let

$$x_N = (\|f\| - \|f\|_{E_N})/\|f\|_{E_N}.$$

Then

$$x_N \leqslant k[\|f\| - \|f\|_{E_N}]$$

for some constant $k > 0$. Since $\ln(1 + x) - x = O(x^2)$ as $x \to 0^+$, we have for all large N,

$$\begin{aligned}\gamma_N = \ln(1 + x_N) \leqslant 2x_N &\leqslant 2k[\|f\| - \|f\|_{E_N}] \\ &\leqslant 2k\,\|f - s_N\| = O(r_N).\end{aligned}$$

The above inequalities follow from the relations

$$\|f - s_N\| = \|f - (f - m_N)\| \geqslant \|f\| - \|f - m_N\| = \|f\| - \|f\|_{E_N}$$

and $\|f - m_N\| = \|f\|_{E_N}$ if m_N is a best approximant to f from M_N. This completes the proof of the theorem.

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